Simplified probabilistic slope stability design charts for cohesive and $c-\phi$ soils

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ABSTRACT

Design charts to estimate the factor of safety for simple slopes with c-\(\phi\) soils are now available in the literature. However, factor of safety is an imperfect measure to quantify the margin of safety of a slope because nominal identical slopes with the same factor of safety can have different probabilities of failure due to variability in soil properties. In this study, simple circular slip slope stability charts for \(\phi = 0\) soils by Taylor (1937) and c-\(\phi\) soils published by Steward et al. (2011) are extended to match estimates of factor of safety to corresponding probabilities of failure. A series of new charts are provided that consider a practical range of coefficient of variation (COV) for cohesive and frictional strength parameters of the soil. The data to generate the new charts were produced using conventional probabilistic concepts together with closed-form solutions for cohesive soil cases and Monte Carlo simulation in combination with conventional limit equilibrium-based circular slip analyses using the program SVSlope for c-\(\phi\) soil cases. The charts are a useful tool for geotechnical engineers to make a preliminary estimate of the probability of failure of a simple slope without running Monte Carlo simulations.

Keywords: Slope stability; Probabilistic analysis; Monte Carlo simulation; Design chart; Spatial variability

1 INTRODUCTION

Slope stability charts are used routinely to estimate the factor of safety of slopes with isotropic, homogeneous soil properties and simple geometry. Taylor (1937) published design charts to calculate the factor of safety for simple homogeneous slopes in clays with single-value undrained shear strength. These charts are found in various forms in geotechnical engineering text books. In the same publication (1937), Taylor presented charts to compute the factor of safety for simple slopes with cohesive-frictional (c-\(\phi\)) shear strength soils. These charts have the disadvantage that they require an iterative procedure to determine the factor of safety. Since Taylor’s work, other researchers have developed slope stability charts for simple slopes with c-\(\phi\) soils. Michalowski (2002) used a kinematic approach of limit analysis with a log-spiral failure mechanism while Baker (2003) and Steward et al. (2011) used conventional limit equilibrium circular slip analyses to produce their charts. All of these methods are deterministic and the resulting charts can be shown to
give the same factor of safety for the same slope and soil properties. The advantage of the work by Steward et al. (2011) is that they produced a single chart for c-\(\phi\) soils which does not require an iterative approach to calculate the factor of safety.

A shortcoming of all these design charts is that an appreciation of the probability of failure of the slope cannot be made. For example, if the soil properties are treated as random variables then it is possible that two slopes with nominally identical soil properties and the same slope geometry can have different probabilities of failure because of differences in variability of the soil properties. In fact, the assessment of probability of slope failure is further complicated because soil properties also have spatial variability.

The influence of the random variability of soil strength parameters on probability of failure of slopes using conventional limit equilibrium slip circle analysis has been explored by Li and Lumb (1987), Chowdhury and Xu (1993), Low et al. (1998) and Hong and Roh (2008). These researchers considered slopes having one or more isotropic, homogeneous soil layers with random strength values described by a single cumulative distribution function. The influence of spatial variability of soil properties on probability of failure using conventional limit equilibrium slope stability analyses has also been investigated by Christian et al. (1994), El-Ramly et al. (2002), Low et al. (2007), Cho (2010) and Wang et al. (2010). Important contributions to the influence of spatial variability of soil properties on stability of slopes have been made by Griffiths and Fenton (2004), Griffiths et al. (2009) and Huang et al. (2010) using the random finite element method (RFEM). All of the above prior work has improved understanding of the quantitative and qualitative influence of the random and spatial distribution of soil strength properties on the margin of safety of slopes in probabilistic terms. However, production of design charts for the estimate of probability of failure of simple slopes with c-\(\phi\) soils and conventional slip circle geometry has not been attempted.

The primary objective of the current study was to develop a series of design charts for simple slopes that combine quantitative estimates of the conventional factor of safety with matching estimates of the probability of failure considering random values of cohesive and frictional shear strength components having lognormal distributions. Two series of design charts are presented.
The first chart is an extension to the Taylor’s chart for purely cohesive soil cases (i.e. undrained shear strength parameters $\phi = \phi_u = 0$, $c = s_u$ and total unit weight $\gamma$). The resulting design chart is equivalent to a chart published by Griffiths and Fenton (2004) for the case of variability in undrained shear strength only. The current study compliments this earlier work by including a useful closed-form expression for calculation of probability of failure for slopes with random values of undrained shear strength and unit weight.

The second series of design charts are for $c$-$\phi$ soils. They were developed from results of Monte Carlo simulation using the probabilistic circular slip slope stability analysis option in the commercially available SVSlope software package (Fredlund and Thode 2011). Together, the two series of charts cover soil slope cases with cohesive and cohesive-frictional soils. Thus this paper provides a convenient single reference for geotechnical engineers to make estimates of the conventional factor of safety and probability of failure for idealized slopes with simple geometry and a wide range of soil properties.

The paper also investigates the influence of cross-correlation of strength parameters and spatial variability on estimates of probability of failure taken from the design charts.

2 SLOPE STABILITY DESIGN CHARTS FOR COHESIVE SOILS

2.1 General

The charts developed for purely cohesive soil cases in this paper use the factor of safety ($F_s$) computed from Taylor’s chart as the independent (input) parameter. The factor of safety is calculated as:

$$F_s = \frac{s_u}{\gamma H N_s}$$

[1]
where \( s_u \) is undrained shear strength, \( \gamma \) is total unit weight, \( H \) is the height of slope and \( N_s \) is a stability number which depends on the slope angle (\( \alpha \)) and depth factor (\( D \)) where \( DH \) is the depth from slope crest to a firm stratum (Figure 1). Slope angle \( \alpha \) and height \( H \) are considered to be deterministic.

The following text shows the details leading to a general expression that is used later to calculate the probability of failure for the case of \( s_u \) as a random variable with lognormal distribution and \( \gamma \) as a constant value or to consider both \( s_u \) and \( \gamma \) as uncorrelated random variables with lognormal distribution.

From probability theory a random variable \( Z \) with a lognormal distribution will have a probability of failure \( P_f \) that can be expressed as:

\[
P_f = P[Z < a] = \Phi \left( \frac{\ln a - \mu_{\ln Z}}{\sigma_{\ln Z}} \right)
\]

where, \( \Phi \) is the cumulative standard normal distribution function, and \( \mu_{\ln Z} \) and \( \sigma_{\ln Z} \) are the mean and standard deviation of the normally distributed random variable \( \ln Z \). In this development, \( Z \) is the factor of safety \( F_s \) defined by Equation 1. If \( s_u \) and \( \gamma \) in Equation 1 are defined as uncorrelated lognormal distributed random variables with mean values of \( \mu_{s_u} \) and \( \mu_{\gamma} \), respectively, and \( N_s \) and \( H \) are constant values, then the mean value and standard deviation of logarithmic values of \( F_s \) can be calculated as follows (Ang and Tang 1984):

\[
\mu_{\ln F_s} = \mu_{\ln s_u} - \mu_{\ln \gamma} - \ln H N_s
\]

\[
\sigma_{\ln F_s} = \sqrt{\sigma_{\ln s_u}^2 + \sigma_{\ln \gamma}^2}
\]

Here, \( \mu_{\ln s_u} \) and \( \mu_{\ln \gamma} \) are mean values of \( \ln s_u \) and \( \ln \gamma \), respectively, and \( \sigma_{\ln s_u} \) and \( \sigma_{\ln \gamma} \) are their corresponding standard deviations. These parameters can be calculated as follows:
Parameters $\text{COV}_{su}$ and $\text{COV}_\gamma$ are coefficients of variation of variable $s_u$ and $\gamma$, respectively. Recall that coefficient of variation is the ratio of standard deviation to mean value. The coefficient of variation of $F_s$ is due only to the variability in uncorrelated random variables $s_u$ and $\gamma$, and is calculated as:

$$COV_{F_s} = \sqrt{COV_{su}^2 + COV_\gamma^2}$$  \[9\]

Algebraic manipulation leads to the following expanded general expression for Equation 2 for $a = F_s = 1$:

$$P_f = p[F_s < 1] = \Phi \left( \ln \left( \frac{1 + COV_{su}^2}{1 + COV_\gamma^2} \frac{1}{F_s} \right) \right) \left( \ln \left( 1 + COV_{su}^2 \right) \left( 1 + COV_\gamma^2 \right) \right)^{1/2}$$  \[10\]

Here, $\bar{F}_s$ in the denominator is the mean factor of safety computed using mean values of $s_u$ and $\gamma$ as follows:
\[ F_S = \frac{\mu_{su}}{\mu_H N_s} \]  

[11]

Equation 10 was used to generate the design curves in Figure 2 using the NORMSDIST function for \( \Phi \) in Excel. For the case of variability in \( s_u \) only, then \( \text{COV}_\gamma = 0 \) in Equation 10 and calculated probabilities of failure are the same as previously reported by Griffiths and Fenton (2004). If a reader uses a characteristic value for \( s_u \) that is less than the mean value, \( \mu_{su} \), then the resulting deterministic estimate of \( F_s \) will be less and the corresponding probability of failure \( (P_f) \) will be greater than those values shown in the chart by unquantifiable amounts.

2.2 Stability charts for cohesive soils

The solid curves plotted in Figure 2 show results of calculations using Equations 9 and 10 for a range of coefficient of variation for \( F_s \) that captures the spread in both \( s_u \) and \( \gamma \) values. To use this chart the mean factor of safety is computed using Equation 11 with \( N_s \) taken from Taylor’s Chart (Figure 1). The quantity \( \text{COV}_{F_s} \) is computed using the coefficients of variation for \( s_u \) and \( \gamma \) as shown in the figure.

It can be seen that as the mean factor of safety, \( F_s \), increases for any constant level of variability in \( s_u \) and \( \gamma \), the probability of failure decreases, which is expected. Also, for \( F_s > 1 \), increasing the spread in soil parameter values increases the probability of failure. Interestingly, for \( F_s < 1 \), increasing \( \text{COV} \) of the soil shear strength and/or unit weight decreases the probability of failure for values of \( \text{COV}_{F_s} < 1 \). The explanation for this behaviour is that when \( F_s \) is greater than one, the ratio inside \( \Phi \) in Equation 10 always increases from a negative value for \( \text{COV}_{F_s} = 0.1 \) to a positive value for \( \text{COV}_{F_s} = 8 \). However, when \( F_s \) is less than one, the ratio decreases or increases depending on the value of \( F_s \) but is always positive. From a practical point of view, slopes with \( F_s < 1 \) are considered to have failed regardless of the corresponding computed probability of failure \( (P_f) \) value and the behaviour noted above is of academic interest only.

Based on recommendations by Phoon and Kulhawy (1999), a reasonable range for the coefficient of variation for \( s_u \) is \( \text{COV}_{su} = 0.1 \) to 0.5 and for \( \gamma \) is \( \text{COV}_\gamma \leq 0.1 \). Hence, curves of practical interest
are between COV_{Fs} = 0.1 and 0.5 in Figure 2 corresponding to the shaded region. The solid lines and the dashed lines in Figure 2 show the differences in computed probability of failure with and without considering the small contribution of variation in \( \gamma \) to computed probabilities of failure. For the curves with COV_{FS} < 1 there is a small difference between solid and dashed lines, but for COV_{FS} > 1 the differences are not visibly detectable. Hence, for practical purposes the solid lines in Figure 2 can be used for both cases. It should be noted that ignoring variability in soil unit weight leads to the same chart published by Griffiths and Fenton (2004) (dashed curves in Figure 2). They used the same probability theory described here together with the assumption of a lognormal distribution for \( s_u \) only.

In addition, a slope stability problem should be treated as a system of potential slip surfaces instead of calculating probability of failure for the most critical slip surface (e.g. Chowdhury and Xu 1995). In such a system, probability of failure depends on the probability of failure of each slip surface and also the correlation between the probabilities of failure of different slip surfaces. However, for homogenous cohesive soil slopes, the system probability of failure is identical to the probability of failure of the critical slip surface provided that the probability of failure along different slip surfaces is highly correlated (the case here).

The accuracy of the design chart in Figure 2 was confirmed by comparing a selection of chart values with the results of Monte Carlo simulation runs using the conventional Simplified Bishop’s Method of analysis in the SVSlope software package (Fredlund and Thode 2011). Probability of failure values obtained using the SVSlope software were slightly higher (a few percent) than corresponding values from Figure 2 but this can be ascribed to differences in the calculation of probability of failure. The Floating Method is used in the numerical computation of probability of failure and is explained later in the paper. The advantage of the approach leading to Figure 2 is that the probability of failure can be computed easily using the closed-form expression described by Equation 10.

Also shown on Figure 2 is an example for the case of \( \bar{F}_s = 1.5 \) and COV_{Fs} = 0.5; this is essentially the same example given by Griffiths and Fenton (2004). This combination gives a probability of failure of 26%. In practice, a (mean) factor of safety of 1.5 would not be expected to match a
probability of failure as high as 26%; indeed, experienced geotechnical engineers would anticipate no probability of failure. An implication of this outcome is the possibility that actual point variability in soils for the simple cases considered here is less than COV = 0.5. Another explanation is that spatial variability of soil with otherwise the same statistical properties may modify the slope margin of safety in probabilistic terms. The influence of spatial variability on predicted probabilities of failure in Figure 2 is investigated later in the paper.

3 COHESIVE-FRICTIONAL SOILS

3.1 General

For simple slopes with cohesive-frictional soils an iterative procedure is necessary to calculate factor of safety using Taylor’s stability chart (Taylor 1937). The chart proposed by Steward et al. (2011) has the advantage that the critical slope factor of safety can be computed without iteration (Figure 3). They used the Slope/W slope stability software package (Geo-Slope Ltd. 2012) to generate the data for their chart. Their design chart also identifies the type of failure circle (not shown here). The failure circles are mostly shallow toe circles; only for shallow slopes with low strength parameters do the critical slip circles move below the toe elevation. In the Steward et al. (2011) chart the input parameters are \( c/(\gamma H \tan \phi) \) and slope angle \( \alpha \). The output parameters are \( c/(\gamma H F_s) \) and \( \tan \phi/F_s \). If the degree of mobilization of both strength components is assumed equal at failure, as was done in the development of their chart and in the numerical calculations to follow, then the factor of safety in the normalized cohesive and frictional strength terms is the same.

In the current study, the SVSlope software package (Fredlund and Thode 2011) was used to carryout circular slip (Simplified Bishop’s Method) analyses together with the Floating Method option for probabilistic analyses (described later). A series of program runs were first used to confirm that the Steward et al. (2011) chart values were accurate and that both the Slope/W and SVSlope software packages gave the same critical factor of safety and the same probability of failure for the same input parameters. For factors of safety greater than one (values of practical
interest) this agreement was verified. There were minor discrepancies for some cases when the factor of safety was less than one, but this was less of a practical concern.

The matrix of computer runs was based on a range of values \( \mu_c/(\mu_r H \tan\mu_\phi) \) (where mean values of \( c, \phi \) and \( \gamma \) are expressed as \( \mu_c, \mu_\phi, \mu_\gamma \) respectively) and slope angle \( \alpha \) from 10 to 90 degrees to cover the range in the original Steward et al. (2011) chart. For each combination of these parameters, the analyses were repeated for a range of coefficients of variation for \( c \) and \( \phi \) (\( \text{COV}_c \) and \( \text{COV}_\phi \), respectively). Both variables were assumed to be lognormal distributed and uncorrelated. Based on recommendations by Phoon and Kulhawy (1999) the range of COV for the two strength parameters was taken as \( \text{COV}_c = 0.1 \) to 0.5 and \( \text{COV}_\phi = 0.1 \) to 0.2. They note that \( \text{COV}_\gamma < 0.1 \) and hence in the analyses to follow variability in \( \gamma \) is ignored (\( \text{COV}_\gamma = 0 \)). The number of Monte Carlo simulations for each case was 4500. This number was calculated automatically by the SVSlope software based on the number of variables (two in this study) and to ensure that repeating the same number of simulations would give the same probability of failure at a 90% confidence level. This condition was verified independently by the writers by repeating a number of trial cases with number of simulation runs up to 30000. The probability of failure from all Monte Carlo simulations was computed as the number of factors of safety less than one divided by the total number of Monte Carlo simulations.

### 3.2 Example results for \( c-\phi \) soils

**Figure 4** shows an example of numerical results for \( \mu_c/(\mu_r H \tan\mu_\phi) = 0.2 \) and \( \mu_\phi = 30 \) degrees. This figure is general for different combinations of \( \mu_c, \mu_\gamma \) and \( H \) as long as \( \mu_c/(\mu_r H \tan\mu_\phi) = 0.2 \). The (deterministic) mean values of \( F_s \) that appear on the horizontal axis can be taken directly from the Steward et al. (2011) chart. As before, the values for \( \mu_c, \mu_\phi \) and \( \mu_\gamma \) are mean values which are the best estimate of each soil property. The curves in the figure show the influence of different combinations and magnitude of coefficients of variation for the two strength parameters (\( c \) and \( \phi \)). The value of \( \text{COV}_\phi \) has been capped at 0.2 consistent with the typical upper range value reported by Phoon and Kulhawy (1999). The general trends and shape of the \( P_r - F_s \) curves are familiar from the cohesive soil design curves presented earlier (**Figure 2**). As before, for each mean factor of
safety greater than one computed deterministically, the probability of failure increases as the COVs of the strength parameters increase. Also shown on the plot is the example case for $F_s = 1.5$ and COV$_c = 0.5$. In this plot the predicted probability of failure is $8\%$. This estimate may still be considered high based on experience, but it is much lower than the value of $26\%$ for the same mean factor of safety in Figure 2. However, Figure 4 is not general and hence the lower probability of failure noted here is not always the case. For example, Figure 5 shows numerical results for the probability of failure for a range of $\mu_c/(\mu_H\tan\mu_\phi)$ values and for maximum spread in $\phi$ and c strength component values (i.e. COV$_\phi = 0.2$ and COV$_c = 0.5$). It can be noted that the curves are progressively more truncated at the low end of $F_s$ as the magnitude of $\mu_c/(\mu_H\tan\mu_\phi)$ increases. Each truncation point corresponds to the maximum slope angle $\alpha = 90$ degrees that can be computed with $\mu_c \geq 0$ and $\mu_\phi = 30$ degrees in the Steward et al. (2011) chart. Figure 5 also shows that for $F_s = 1.5$ and typical maximum variability in c and $\phi$, the probability of failure is about $18\%$, which again appears high for a slope with $F_s = 1.5$ based on experience.

Results of calculations presented in Figures 4 and 5 are useful since they provide insight into the relationship between probability of failure, mean factor of safety and normalized strength parameter $\mu_c/(\mu_H\tan\mu_\phi)$ for an example with fixed variability in c and $\phi$ and $\gamma$ input parameters. However, they are not general and hence their practical utility for design is limited. In the next section, general design charts are presented using all results from Monte Carlo simulation of simple slopes with c-$\phi$ soils.

### 3.3 Simplified probabilistic slope stability design charts for c-$\phi$ soils

Figures 6 through 11 are simplified probabilistic stability design charts for the general case of cohesive-frictional (c-$\phi$) soils with $\mu_\phi = 20, 25, 30, 35, 40$ and 45 degrees. The input parameters to compute the conventional factor of safety $F_s$ are $\mu_c/(\mu_H\tan\mu_\phi)$ and $\alpha$ as in the Steward et al. (2011) chart. Hence, these charts provide an estimate of the conventional factor of safety for a slope using mean estimates of the slope soil properties. Other values of $\mu_c/(\mu_H\tan\mu_\phi)$ can be interpolated between the long dash lines.
Superimposed on these charts are solid lines corresponding to probabilities of failure ($P_f$) assuming maximum variability in the strength properties based on recommended values by Phoon and Kulhawy (1999) (i.e. $\text{COV}_c = 0.5$ and $\text{COV}_\phi = 0.2$). These curves are taken to 0.01% probability of failure. Silva et al. (2008) reported that $P_f = 0.01\%$ (annual probability of failure) corresponds to an earth dam designed to a factor of safety of 1.5 using above average level of engineering. The short dash lines in the figures show example probability of failure curves using lower estimates of the spread in $c$ and $\phi$ (i.e. $\text{COV}_c = 0.1$ and $\text{COV}_\phi = 0.1$). The spread of each set of these probability curves is less because the COV for $c$ and $\phi$ values is smaller. For these sets of curves, lines corresponding to probability of failure of 1% and 0.01% are very close, therefore only lines corresponding to 1% probability of failure are shown in the charts. The accuracy of Figures 6 through 11 was confirmed for the same ratios of $\mu_c/\mu \tan \mu_\phi$ but using different values of $\mu_c, \mu_\phi$ and $H$ in the SVSlope software.

For simple slope cases with the same slope angle $\alpha$ and $\mu_c/\mu_\phi H$ but other mean friction angles between 20 and 45 degrees, the mean factor of safety can be linearly interpolated with mean friction angle to sufficient practical accuracy as demonstrated by the plot in Figure 12. The logarithm of the corresponding probability of failure also varies smoothly with mean friction angle showing that log-linear interpolation between charts for $P_f$ values is reasonable. The accuracy of the interpolation curves was confirmed using the SVSlope software and a range of other values of $\mu_\phi$, $\text{COV}_c$ and $\text{COV}_\phi$. The arrows in Figure 12 show that for the example case $\alpha = 45$ degrees, $\mu_\phi = 37$ degrees and $\mu_c/\mu_\phi H = 0.05$, the corresponding mean factor of safety is 1.42 and the probability of failure is 7%.

4 DISCUSSION

4.1 Influence of method of analysis

There are two options (Fixed and Floating Methods) in the SVSlope software to find the critical slope and assign a probability of failure. The Fixed Method first locates the critical slip surface based on a conventional deterministic slope stability analysis. Then, the factor of safety of this
single slip circle is recalculated using random values of the input soil parameters (i.e. Monte Carlo simulation). The probability of failure is then computed as the fraction of outcomes with factor of safety less than one. In the Floating Method, each trial slip surface is reanalyzed using random variables of the soil input parameters. The probability of failure is then calculated as before using all computed factors of safety from all simulations on all trial circles. This approach is computationally more expensive but it is more general and in principle should give a more accurate estimate of the true critical slip surface based on maximum probability of failure. Hence, the Floating Method was used to generate the data to construct the design charts for c-ϕ soil cases in this study.

It can be noted that Fixed and Floating methods were found to give the same mean factor of safety (Equation 1) and probability of failure (Equation 10) for the case of cohesive soil slopes with no spatial variability in soil parameters (i.e. the values in Figures 1 and 2).

For cohesive and c-ϕ soil cases when spatial variability is considered, results using the Fixed Method were found to give lower probabilities of failure than the Floating Method. However, the relative magnitude of the difference in $P_f$ using both methods is less for the c-ϕ soil cases than the cohesive soil cases. These observations are consistent with the remarks made by Cho (2010).

4.2 Influence of cross correlation between c and ϕ

The cross correlation ($\rho$) between strength parameters is considered to incorporate the dependence between c and ϕ. A negative correlation between c and ϕ is reported in literature (Yucemen et al. 1973; Lumb 1970; Cherubini 2000) and implies that low values of friction angle are associated with high values of cohesion and vice versa. The uncertainty in the calculated shear strength is smaller when negative correlation between c and ϕ is considered rather than the combined uncertainty in the two parameter values used to model the shear strength.

Figure 13 shows probability of failure versus mean factor of safety for the example slope with $\mu_c/(\mu_c \tan \mu_\phi) = 0.2$ and COV$_c = 0.5$ and COV$_\phi = 0.2$ and $\mu_\phi = 30$ and different correlation coefficients $-1 < \rho < 0$ ($\rho = 0$ means that c and $\phi$ are uncorrelated parameters and $\rho = -1$ means that c...
and $\phi$ are highly correlated parameters). In Figure 13, the curve with $\rho = 0$ is the same example curve shown in Figure 4 which has probability of failure of 8% for the corresponding mean factor of safety equal to 1.5. It can be seen in Figure 13 that for $\bar{F}_s \geq 1.08$ the probability of failure $P_f$ decreases as the correlation coefficient $\rho$ becomes more negative. The implication is that the probabilities of failure from Figures 6 to 11 are conservative for design.

4.3 Influence of spatial variability of c and $\phi$

Spatial variability of soil strength parameters can be considered in SVSlope software using the Distance Option (sampling distance). This option uses the Local Average Subdivision method described by Fenton and Vanmarcke (1990) to model soil random fields in probabilistic slope stability analysis. Sampling distance in the SVSlope software is equal to the scale of fluctuation. The inset drawing in Figure 14 shows a constant distance (equal to scale of fluctuation) ($\Delta L$) located along the arc length (L) of the circular slip. If distance $\Delta L$ is equal to or greater than the arc length then the same soil properties taken from random sampling are assigned to all slices during each circular slip analysis. This is the default case used to generate the previous design charts. If the distance ($\Delta L$) is less, new random soil values using the same soil property statistics for the control case are assigned to each $\Delta L$ segment. These random values are assigned to all slices whose base mid-point falls within the same $\Delta L$ segment (solid black circles). The correction method used in SVSlope to treat a truncated $\Delta L$ segment located at the end of the arc length has been taken from Vanmarcke (1983). The method is described in the software manual and for brevity is not repeated here.

The scale of fluctuation is taken as a twice the spatial correlation length $\theta$ (El-Ramly et al. 2003). A large spatial correlation length value implies that the soil property is highly correlated over that distance, resulting in a smooth variation within the soil profile. Conversely, a small value indicates that the fluctuations of the soil property are large (El-Ramly 2001). Soil properties are typically more variable in the vertical direction than the horizontal direction. El-Ramly (2001) and Phoon and Kulhawy (1999) suggested values of autocorrelation length between 10 to 40 m and 1 to 3 m in the horizontal and vertical directions, respectively. Figure 14 shows the influence of spatial variability of soil strength parameters on probability of failure for an example slope. The upper
curve represents the idealized condition of no spatial variability in soil shear strength and is general. The dashed curves are for the case with $\alpha = 27$ degrees, $H = 10$ m, $D = 2$ and $\mu_r = 17$ kN/m$^3$ and different scale of fluctuation. The figure shows that as scale of fluctuation decreases (and spatial variability of shear strength increases) the probability of failure decreases and all dashed curves are sensibly less than the control case for $\bar{F}_s > 1$. Hence, for this case, Figure 2 can be considered to give upper-bound estimates of probability of failure and is thus conservatively safe for design. However, the potential influence of spatial variability on probability of failure will decrease as the height of the slope decreases because autocorrelation lengths are independent of slope height and geometry.

Cho (2007) reported two examples of probabilistic circular slip analysis using layered c-$\phi$ soils with high factors of safety ($\bar{F}_s > 1.5$). These examples also gave decreasing probabilities of failure with decreasing scale of fluctuation consistent with the trends in Figure 14 for the same range of factor of safety.

Figure 15 shows the influence of normalized scale of fluctuation on probability of failure for simple slope geometry and c-$\phi$ soil properties identified in the caption. It can be seen that for $\bar{F}_s > 1$, the probability of failure decreases with decreasing scale of fluctuation. The upper bound curve ($\Theta = \infty$) is the case when $\Delta L \geq L$ (spatial variability is not considered). The lower limit on $P_f$ occurs when each slice is resampled. In the limit of a very small slice width ($\Delta x$) the curves converge to the deterministic value which is of no practical value. Furthermore, only those curves that correspond to a scale of fluctuation at least twice the autocorrelation length of the soil properties are of practical value as noted earlier.

### 4.4 Random finite element modeling

An alternative approach to investigation of the influence of spatial variation on probability of failure of slopes is the use of the random finite element method (RFEM). Important references on this approach are given in the introduction and in the textbook by Fenton and Griffiths (2008). The method involves creation of a finite element mesh with each element assigned a random property value. The distribution of property values may be random or spatially distributed. The soil
is assumed to behave as a linear-elastic plastic material and each finite element mesh with an assignment of soil properties is taken as one realization. The finite element program redistributes soil strength to satisfy the failure criterion and global equilibrium under gravity loading for each realization. If the number of iterations to meet convergence criteria is greater than a prescribed number, then the realization is assumed to represent failure. As in the probabilistic slope stability method used in this paper, the number of realizations that fail divided by the total number of simulations is the probability of failure.

At the time of the current study, RFEM slope stability methods are research tools and have yet to be adopted by geotechnical engineers in practice and implemented in commercial slope stability programs. However, the advantage of the RFEM method is that geometry of the critical failure mechanism is not constrained, as is the case for conventional circular slip slope stability methods. Each simulation will seek out the weakest path. This means that estimates of probability of failure for the same nominal slope may be different between the RFEM slope stability approach and the probabilistic slope stability approach adopted in the current study.

Griffiths and Fenton (2004, 2009, 2010) have shown that conventional probabilistic slope stability analyses of the type used in the current study for cohesive soils may give non-conservative design outcomes in some cases if spatial variability in the soil properties is not considered. Griffiths and Fenton (2004) carried out random finite element method modelling of cohesive soil slopes for the case when spatial variability in the soil properties is considered. They showed that when spatial variability was included in the distribution of $s_u$ for cases with deterministic $F_s > 1.37$ and $\text{COV}_{s_u} = 0.5$, the probability of failure was less when spatial variability was included. The same is true in Figure 14 for $F_s > 1$ which was developed using the same slope geometry and soil properties as Griffiths and Fenton (2004). The difference in the breakpoint value of deterministic $F_s = 1.37$ in their work and $F_s = 1$ (Figure 14) is related to the difference in the way the critical mechanisms are identified using these two methods and how spatial variability of soil properties is assigned to these two very different approaches to probabilistic slope stability analysis. In many cases, the practical minimum (mean) factor of safety for design of slopes is 1.5. Hence, the differences noted above can be argued to be more of academic interest than practical concern.
5 CONCLUSIONS

The slope stability design charts developed in the current study apply to slopes with simple geometry and random values of soil shear strength parameters $s_u$, $c$, $\phi$ and $\gamma$.

The utility of the first chart (Figure 2) is that it allows the well-known Taylor (1937) chart (Figure 1) for cohesive soil slopes to be extended to include an estimate of the probability of failure for each conventional mean factor of safety estimate. The chart in Figure 2 can be used for soils having a range of coefficient of variation in $s_u$ and $\gamma$ matching the range reported in the literature. Alternatively, the probability of failure can be calculated directly using the closed-form expression (Equation 10) and values for the uncorrelated mean and coefficient of variation for the lognormal distributions for $s_u$ and $\gamma$.

The second series of charts (Figures 6 through 11) allow the conventional factor of safety to be estimated for simple slopes with cohesive-frictional (c-$\phi$) soil strength and simultaneously estimate the corresponding probability of failure. These charts are presented for the case of maximum typical estimates of coefficient of variation in $c$ and $\phi$. In this paper charts for $\mu_\phi = 20, 25, 30, 35, 40$ and 45 degrees and $\mu_c \geq 0$ are presented. For other friction angle values within this range, the factor of safety can be interpolated between charts.

Effects of cross correlation between strength parameters and spatial variability of strength parameters on probability of failure are also considered in this paper. For $\bar{F}_s > 1.08$ and a negative cross correlation between strength parameters $c$ and $\phi$ (the usual case) probabilities of failure were less than values for the idealized case of uncorrelated strength parameters. There is evidence in the related literature that spatial variability in soil strength properties can also influence the probability of failure for a soil slope. The probabilistic slope stability analyses of the type carried out in this study showed that considering soil spatial variability will decrease the probability of failure of a slope with $\bar{F}_s > 1$ when compared to the nominal identical simple slope case with point (random) variability in soil shear strength values. Hence, the probabilities of failure using the design charts in
this paper may be considered to be conservative estimates (i.e. safe) for the analysis and design of simple slopes.

It should be noted that estimating the variability of any soil property is difficult in practice and this challenge is compounded when estimates of spatial variability are attempted for real world cases. Consequently, the charts presented in this paper are useful for preliminary upper-bound estimates of the conventional factor of safety and probability of failure of simple slopes with idealized soil conditions and simple failure geometries (i.e. circular). In real world cases failure surfaces are often irregular and therefore difficult to prescribe, and strongly coupled to spatial heterogeneity beyond the scale of fluctuations in soil properties (e.g. layered soils). For cases involving layered soils and more complicated surface geometry, probabilistic slope stability analyses can be carried out using available commercial software packages or spreadsheet-based methods (e.g. Low et al. 1998, 2007; Low 2003; Wang et al. 2010). Probabilistic slope analyses with layered soils introduce additional challenges with respect to the number of searches to find the most critical mechanism. Strategies to reduce computational effort for these cases can be found in the papers by Ching et al. (2009, 2010) and Wang et al. (2010).

An important contribution of this paper is that it provides a link between probabilistic slope stability analyses and the classical factor of safety approach which up until now has been difficult to understand by many geotechnical engineers. By using simple cases this link is not obscured by more complicated real world cases, complex mathematics and unfamiliar terminology. The conservative outcomes using the idealized probabilistic approaches in this paper with respect to classical factor of safety expectations are explained. The paper alerts the reader to the influence of real world random and spatial variability of soil properties and large-scale heterogeneity due to layering on computed probability of failure of slopes.

Finally, the charts presented in this paper are also useful for geotechnical engineers to check computer-based probabilistic slope stability analysis outcomes against baseline cases.
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REFERENCES


Figure 1. Taylor’s slope stability chart for cohesive soils (Taylor 1937)
Figure 2. Probability of failure ($P_f$) versus (deterministic) mean factor of safety ($\bar{F}_s$) for cohesive soil slopes with lognormal distribution of undrained shear strength ($s_u$) and unit weight ($\gamma$). Note: Shaded region is the practical range. Dashed lines are for cases with $\text{COV}_\gamma = 0$. 

$$\bar{F}_s = \frac{s_u}{\mu_x H N_s}$$

$$\text{COV}_{\bar{F}_s} = \sqrt{\text{COV}_{s_u}^2 + \text{COV}_{\gamma}^2}$$
Figure 3. Slope stability design chart for cohesive-frictional soils (after Steward et al. 2011).
Figure 4. Probability of failure ($P_f$) versus (deterministic) mean factor of safety ($\bar{F}_s$) for value of $\mu_c/(\mu_R\tan\phi) = 0.2$ with $\phi = 30$ degrees and a range of COV for strength parameter values.
Figure 5. Probability of failure ($P_f$) versus (deterministic) mean factor of safety ($\bar{F}_s$) for $\text{COV}_c = 0.5$ and $\text{COV}_\phi = 0.2$ and a range of $\mu_c/(\mu_\gamma H \tan \mu_\phi)$
Figure 6. Probabilistic slope stability design chart for $\mu_\phi = 20$ degrees and $\text{COV}_c = 0.5$, $\text{COV}_\phi = 0.2$ and $\text{COV}_c = \text{COV}_\phi = 0.1$
Figure 7. Probabilistic slope stability design chart for $\mu_\phi = 25$ degrees and COV$_c = 0.5$, COV$_\phi = 0.2$ and COV$_c = \text{COV}_\phi = 0.1$. 
Figure 8. Probabilistic slope stability design chart for $\mu_\phi = 30$ degrees and COV$_c = 0.5$, COV$_\phi = 0.2$ and COV$_c = COV_\phi = 0.1$
Figure 9. Probabilistic slope stability design chart for $\mu_\phi = 35$ degrees and $\text{COV}_c = 0.5$, $\text{COV}_\phi = 0.2$ and $\text{COV}_c = \text{COV}_\phi = 0.1$
Figure 10. Probabilistic slope stability design chart for $\mu_\phi = 40$ degrees and $\text{COV}_c = 0.5$, $\text{COV}_\phi = 0.2$ and $\text{COV}_c = \text{COV}_\phi = 0.1$
Figure 11. Probabilistic slope stability design chart for $\mu = 45$ degrees and $\text{COV}_c = 0.5$, $\text{COV}_\phi = 0.2$ and $\text{COV}_c = \text{COV}_\phi = 0.1$. 

$\mu_c/\left(\gamma H \tan \mu\right) = 0, 0.01, 0.03$
Figure 12. (Deterministic) mean factor of safety and probability of failure from Figures 6 to 11 versus mean friction angle $\mu_\phi$ for a range of $\mu_c/(\mu_c H)$ for slopes with $\alpha = 45$ degrees and $\text{COV}_c = 0.5$, $\text{COV}_\phi = 0.2$
Figure 13. Probability of failure ($P_f$) versus (deterministic) mean factor of safety ($\bar{F}_s$) for $COV_c = 0.5$ and $COV_\phi = 0.2$ and with $\mu_\phi = 30$ degrees for negative values of cross correlation between $c$ and $\phi$.
Figure 14. Example of influence of sampling distance on probability of failure ($P_f$) for slope for $\alpha = 27$ degrees, $H = 10$ m, $D = 2$, $\mu_\gamma = 17$ kN/m$^3$, $COV_{su} = 0.5$ and $COV_\gamma = 0$
Figure 15. Probability of failure ($P_f$) versus (deterministic) mean factor of safety ($\bar{F}_s$) for $\text{COV}_c = 0.5$ and $\text{COV}_\phi = 0.2$ with $\mu_\phi = 30$ degrees for different normalized scale of fluctuation ($\Theta = \Delta L/H$)